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THE MILNOR NUMBER FOR CURVES ON TORIC SURFACES

by

Jan Arthur Christophersen

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It is proved in [D] that the topological type of an isolated curve singularity on a toric surface depends only on its Newton filtration. In this note we give as a corollary of Kouchnirenko's theorem I in [K] a similar formula for the Milnor number of singularities defined by one analytic function on a normal affine two dimensional toric variety. I would like to thank James Damon for his helpful remarks.

1. The toric surface as quotient singularity

Any complex normal affine toric surface is isomorphic to one of the type $X = \mathrm{Spec}(\mathbb{C}[\Lambda])$ where $\Lambda = \sigma \cap \mathbb{Z}^2$, σ is a rational polyhedral cone in $\mathbb{R}^2_+$ and $\mathbb{C}[\Lambda]$ is the semigroup ring of Λ over $\mathbb{C}$. (See [O])

Let σ be generated by $M_1 = (m_1, n_1)$ and $M_2 = (m_2, n_2)$, i.e. $M_1, M_2 \in \mathbb{Z}^2_+$, $\gcd(m_i, n_i) = 1$, $\det(M_1, M_2) > 0$ and $\sigma = \{\alpha_1 M_1 + \alpha_2 M_2 \mid \alpha_i \in \mathbb{R}_+\}$. It is well known that $X = \mathbb{C}^2/G$ where G is a finite cyclic group acting freely outside the origin. One way of seeing this is as follows: Let $G = \{\alpha_1 M_1 + \alpha_2 M_2 \mid \alpha_i \in \mathbb{R},\ 0 \leq \alpha_i < 1\} \cap \mathbb{Z}^2$. Then G is a cylic group of order $d = \det(M_1, M_2)$; the group addition is vector addition modulo $\langle M_1, M_2 \rangle$.

Define two linear maps $\nu_1, \nu_2 : \mathbb{Z}^2 \to \mathbb{Z}$ by $\nu_1(x) = \det(x, M_2)$, $\nu_2(x) = \det(M_1, x)$. These restrict to $\nu_i : G \to \{0, 1, \ldots, d-1\}$ and induce group isomorphisms $\bar{\nu}_i : G \xrightarrow{\sim} \mathbb{Z}_d$, $\bar{\nu}_i(a) = \nu_i(a) \pmod d$. We have an action of G on $\mathbb{C}[x,y]$; for $a \in G$

$$a.x = e^{\frac{-2\pi i \nu_2(a)}{d}} x$$

$$a.y = e^{\frac{2\pi i \nu_1(a)}{d}} y$$

Let $T: \mathbb{Z}^2 \to \mathbb{Z}^2$ be the linear transformation $T(m,n) = (\nu_1(m,n), \nu_2(m,n))$. It's matrix is $\begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix}$. Of course $\mathbb{C}[\Lambda] \simeq \mathbb{C}[T(\Lambda)] \subseteq \mathbb{C}[x,y]$ and one checks that $\mathbb{C}[T(\Lambda)] = \mathbb{C}[x,y]^G$.

On the other hand, given a cyclic group $G \subseteq GL_2(\mathbb{C})$ and an action on $\mathbb{C}^2$ one can easily construct a cone $\sigma \subseteq \mathbb{R}^2_+$ such that Spec $\mathbb{C}[\sigma \cap \mathbb{Z}^2] = \mathbb{C}^2/G$.

2. The Milnor number of a function on X

Assume f is an analytic function $f: X \to \mathbb{C}$ with an isolated critical point at 0. We know that f has a Milnor fibration [L], i.e. a C^∞ fibration

$$B_\varepsilon \cap X \cap f^{-1}(D^*_\eta) \to D^*_\eta$$

induced by f, where B_ε is a closed regular ball in $\mathbb{C}^e$ with radius $\varepsilon > 0$ and D^*_η is a punctured disc of radius η in $\mathbb{C}$. (Here e denotes the embedding dimension of X.) If F is the typical Milnor fiber we can define $\mu(f) = \mathrm{rk}\, H_1(F)$ to be the Milnor number of the curve $f^{-1}(0)$ at 0.

We are now in the following situation:

$$\begin{tikzcd} \bar{F} \subseteq \mathbb{C}^2, 0 \arrow[d, "\pi"'] \arrow[drr, "\bar{f}"] & & \\ F \subseteq X, 0 \arrow[rr, "f"] & & \mathbb{C}, 0 \end{tikzcd}$$

π is the natural map $\pi: \mathbb{C}^2 \to \mathbb{C}^2/G$.

Here $\bar{f} = f \circ \pi$ and $\bar{F}$ is the Milnor fiber of $\bar{f}$ at the origin.

The embedding $\psi: X \to \mathbb{C}^e$ is given by monomials in x and y generating $\mathbb{C}[x,y]^G$. Call these generators for $\phi_1,\ldots,\phi_e$ and put $w_i = \deg \phi_i$. Then $\phi = \psi \circ \pi: \mathbb{C}^2 \to \mathbb{C}^e$ is the map $\phi(x,y) = (\phi_1(x,y),\ldots,\phi_e(x,y))$. If N is any positive number let

$$B'_{\varepsilon,N} = \{z\epsilon\mathbb{C}^e: \sum_{i=1}^{e} |z_i|^{2b_i} < \varepsilon,\ b_i \cdot w_i = N\}.$$

The Milnor fiber constructed from regular balls in $\mathbb{C}^e$ is diffeomorphic to the Milnor fiber using "weighted" balls of the form $B'_{\varepsilon,N}$ for suitable choice of ε.

Notice that:

1) The unit sphere in $\mathbb{C}^2$ is compact so, $\max\{|\phi_1(x,y)|^2,\ldots,|\phi_e(x,y)|^2: |(x,y)| = 1\}$ exists.

2) $$\sum_{i=1}^{e} |\phi_i(x,y)|^{2b_i} = \sum_{i=1}^{e} |(x,y)|^{2b_i w_i} \cdot \left|\phi_i\left(\frac{(x,y)}{|(x,y)|}\right)\right|^{2b_i}$$
$$= |(x,y)|^{2N} \sum_{i=1}^{e} \left|\phi_i\left(\frac{(x,y)}{|(x,y)|}\right)\right|^{2b_i}$$

It follows that for appropriate ε and N, ϕ maps a regular ball of $\mathbb{C}^2$ onto $X \cap B'_\varepsilon$. This means we can assume that $F = \bar{F}/G$;

hence $\chi(\bar{F}) = (\text{ord } G) \cdot \chi(F)$, where χ is the Euler characteristic. But since we are dealing with curves,
$\chi(\bar{F}) = 1 - \mu(\bar{f})$ and $\chi(F) = 1 - \mu(f)$, so

$$\mu(f) = 1 + \frac{\mu(\bar{f})-1}{d}.$$

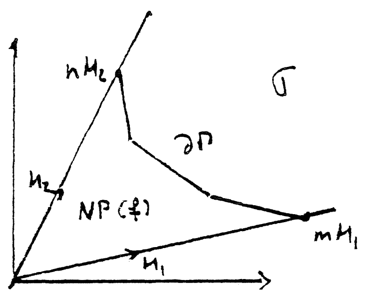
Recall Kouchnirenko's formula for the Milnor number of a plane curve at the origin. A power series $g \in \mathbb{C}[[x_1,x_2]]$ is "commode" if the monomials x_1^m and x_2^n, $m, n \geq 1$, appear in g with non zero coefficients. If m and n are the minimal such numbers, then the

Newton number $\nu(g)$ is defined as $2A-m-n+1$ where A is the area bounded by the Newton polygon $NP(g)$ of g. If g is not "commode" then $\nu(g)$ is defined by $\nu(g) = \sup_{m \in N} \nu(g+\sum x_i^m)$ where the sum is now taken over the variables, a power of which does not appear alone in g. The theorem then states that if g has an isolated critical point at 0 and if g is nondegenerate (see [K], page 7, for the definition) then $\mu(g) = \nu(g)$.

Identifying an analytic $f: X \to \mathbb{C}$ with a power series in the monomials of $\mathbb{C}[\Lambda]$, we can define its Newton polygon as follows. If

$$f = \sum_{(i,j)\in\Lambda} c_{ij} x^i y^j, \text{ let } \Gamma \text{ be the convex hull in } \sigma \text{ of } \cup\{(i,j)+\sigma\}$$

where $(i,j)\in\{(i,j)\,|\,c_{ij}\neq 0\}$. Call f commode if the monomials $(x^{m_1}y^{n_1})^m$ and $(x^{m_2}y^{n_2})^n$ appear with non zero coefficients. If f is commode then define the Newton polygon $NP(f)$ to be the polygon given by the boundary of Γ and the rays through M_1 and M_2.



Define the Newton number as $\nu(f)=2S-m-n+1$ where S is the area bounded by $NP(f)$ and m and n are chosen minimally as above. If f is not commode, then define $\nu(f)$ as in the case of plane curves above.

To make the definitions simple we will say that f is nondegenerate if $\bar{f} = f\circ\pi$ is nondegenerate.

<u>Theorem</u> Let $f: X \to \mathbb{C}$ be an analytic function with an isolated critical point at the origin. If f is nondegenerate, then $\mu(f) = \nu(f)$.

<u>Proof</u>. Each monomial in $\bar{f}$ is just the image of the corresponding monomial in f by the mapping $\mathbb{C}[\Lambda] \tilde{\to} \mathbb{C}[T(\Lambda)] \to \mathbb{C}[x,y]$. So $NP(\bar{f})$ is just the image of $NP(f)$ by

$$T = \begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

If f is commode, $f=(x^{m_1}y^{n_1})^m + (x^{m_2}y^{n_2})^n + \dots$, then the area bounded by $NP(\bar{f})$ is $(\det T)\cdot S = d\cdot S$. Since $T(mM_1)=(md,0)$, $T(nM_2)=(0,nd)$,

$$\mu(\bar{f}) = \nu(\bar{f}) = 2Sd - md - nd + 1$$

Consequently

$$\mu(f) = 1 + \frac{\mu(\bar{f})-1}{d} = 2S - m - n+1 = \nu(f)$$

The same argument obviously holds in the non-commode case. □

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University of Oslo
Inst. of Mathematics
P.B. 1053, Blindern
0316 Oslo 3
NORWAY